IMA Journal of Applied Mathematics (2002) **67**, 551–557

On ϕ_0-boundedness and -stability properties with perturbing Liapunov functional

A. A. SOLIMAN†
Department of Mathematics, Faculty of Sciences, Benha University, Benha 13518, Kalubia, Egypt.

[Received on 25 June 2001]

In this paper we will extend the perturbing Liapunov function method to systems of functional differential equations and discuss ϕ_0-equiboundedness and ϕ_0-equistability properties via the concept of perturbing Liapunov functional.

Keywords: Liapunov functional-ϕ_0-equibounded; uniformly ϕ_0-bounded; ϕ_0-equistability; uniform ϕ_0-stability.

1. Introduction

Let $\mathbb{R}^n$ be Ecludean n-dimensional real space, with any convenient norm $\| \, . \, \|$ and scalar product $(. \, , .) \leqslant \| \, . \, \| \| \, . \, \|$, $\mathbb{R}^+ = [0, \infty)$ and $C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$ denotes the space of continuous mappings $\mathbb{R}^+ \times \mathbb{R}^n$ into $\mathbb{R}^n$. The following definitions (Akpan & Akinyele, 1992) will be needed in the sequel.

DEFINITION 1.1 A proper subset $K \subset \mathbb{R}^n$ is called a *cone* if
(i) $\lambda K \subset K, \lambda \geqslant 0$, (ii) $K + K \subset K$, (iii) $\overline{K} = K$, (iv) $K^\circ \neq \emptyset$, (v) $K \cap (-K) = 0$, where $\overline{K}$ and K° denote the closure and interior of K respectively, and ∂K denotes the boundary of K. The order relation on $\mathbb{R}^n$ induced by the cone K is defined as follows.

Let $x, y \in K$ then $x \leqslant_{K_y} \Longleftrightarrow y - x \in K$, and $x \leqslant_{K_y^\circ} \Longleftrightarrow y - x \in K^\circ$.

DEFINITION 1.2 The set K^* is called the *adjoint cone* if

$$K^* = \{\phi \in \mathbb{R}^n : \phi, x \geqslant 0\}, \quad x \in K,$$

satisfies properties (i)–(v) of Definition 1.1.

DEFINITION 1.3 A function $g : D \to \mathbb{R}^n, D \subset \mathbb{R}^n$, is called *quasimonotone* relative to the cone K, if whenever $x, y \in D$ and $y - x \in \partial K$, there exists $\phi_0 \in K_0^\star$ such that $(\phi_0, y - x) = 0$ and $(\phi_0, g(y) - g(x)) \geqslant 0$.

Consider the system of functional differential equations

$$x' = f(t, x_t), \quad x_{t_0} = \psi, \tag{1.1}$$

†Present address: Department of Mathematics, Faculty of Teachers, Al-Jouf, Skaka, PO Box 269, Kingdum of Saudi Arabia.

where $f \in C[J \times C_0, K]$, $\mathbb{R}^n$ is Euclidean n-dimensional real space, $J = [t_0, \infty]$,

$$\wp^n = C[[-r, 0], \mathbb{R}^n], \quad C_0 = \{\phi \in \wp^n : \| \phi \|_0 < \rho\} \quad \text{and} \quad \| \phi \|_0 = \max_{-r \leqslant s \geqslant 0} \| \phi(s) \| .$$

Here $C[J \times C_0, K]$ denotes the space of continuous mappings $J \times C_0$ into K. For $x_t(s) = x(t+s), -r \leqslant s \leqslant 0$, and $x_t(t_0, \psi)$ is a solution of (1.1) with initial values $x_{t_0} = \psi$. Define

$$s_0(\rho) = \{x_t \in C_0 : \| x_t \| < \rho\}.$$

The Liapunov function plays an essential role in determining the zero solution of the system of ordinary of differential equations.

Lakshmikantham & Leela (1976) introduced the method of perturbing the Liapunov function for systems of ordinary differential equations, and Soliman (2002) extended this method to systems of functional differential equations. This method discussed non-uniform properties of solutions of systems of differential equations.

Stability properties of systems of differential equations have been of interest to many authors via perturbing Liapunov function (see Koksal, 1992a,b).The notion of ϕ_0-stability was introduced by (Akpan & Akinyele, 1992), and in (El-sheikh & Soliman, 1995, 2000) the notion was improved and extended to systems of functional differential equations.

The main purpose of this paper is to discuss the notions of ϕ_0-equiboundedness and ϕ_0-equistability of the system (1.1) via the perturbing Liapunov functional of (Soliman, 2002).

Following (Lakshmikantham & Leela, 1969b), we define a Liapunov functional: $V(t, x_t) \in C[J \times C_0, \mathbb{R}^n]$ is Lipschitzain in x_t, and the functional is

$$D^+V(t, x_t) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, x_{t+h}) - V(t, x_t)].$$

The first work dedicated to this method was done by Lakshmikantham & Leela (1976).

The following is somewhat new and is related to a definition of (Akpan & Akinyele, 1992).

DEFINITION 1.4 A solution $x_t(t_0, \psi)$ of (1.1) is said to be ϕ_0-equibounded if there exist positive constants $M(t_0, \delta) > 0$ and $\delta > 0$, such that for $\phi_0 \in K_0^*$

$$(\phi_0, \psi) \leqslant \delta \Longrightarrow (\phi_0, x_t^*) \leqslant M(t_0, \delta).$$

DEFINITION 1.5 (Akpan & Akinyele, 1992). The zero solution of (1.1) is said to be ϕ_0-equistable if for $\epsilon > 0, t_0 \in J$, there exists a positive function $\delta(t_0, \epsilon) > 0$ that is continuous in t_0 such that for $\phi_0 \in K_0^*$

$$(\phi_0, x_t^*) < \epsilon, \quad t \geqslant t_0,$$

provided that $(\phi_0, \psi) < \delta$, where x_t^* is the maximal solution of (1.1).

In the case of uniform ϕ_0-stability, δ is indepentent of t_0.

2. ϕ_0-equiboundedness

In this section, we discuss ϕ_0-boundedness of the systems (1.1) via perturbing Liapunov functional. Now, we define the following class.

DEFINITION 2.1 A function $b(r)$ is said to be belong the class $\mathcal{K}$ if $b \in C[(0,\rho), \mathbb{R}^+]$, $b(0) = 0$, $b(r) \longrightarrow 0$ as $r \longrightarrow 0$, and $b(r)$ is strictly increasing in r.

THEOREM 1 Let $E \subset C_0$ be a compact subset. Suppose that there exist two functionals $V_1(t, x_t) \in C[J \times \overline{E^c}, K]$, $V_2(t, x_t) \in C[J \times s_0^c(\rho), K]$, and there exist two functions $G_1 \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^n]$, $G_2 \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^n]$ with $V_1(t,0) = V_2(t,0) = G_1(t,0) = G_2(t,0) = 0$ such that for $\phi_0 \in K_0^*$

(A$_1$) $V_1(t, x_t)$ is Lipschitzian in x_t and

$$D^+(\phi_0, V_1(t,x_t)) \leqslant G_1(t, V_1(t,x_t)), \quad (t,x_t) \in J \times \overline{E^c}; \tag{2.1}$$

(A$_2$) $V_2(t, x_t)$ is Lipschitzain with respect to x_t and

$$b(\phi_0, x_t^*) \leqslant (\phi_0, V_2) \leqslant a(\phi_0, x_t^*), \tag{2.2}$$

where $a, b \in \aleph$, $\phi_0 \in K_0^*$, and $(t, x_t) \in (J \times s_0^c)$;

(A$_3$) for each $(t, x_t^*) \in J \times s_0^c$,

$$D^+(\phi_0, V_1) + D^+(\phi_0, V_2) \leqslant G_2(t, V_1(t,x_t) + V_2(t,x_t)); \tag{2.3}$$

(A$_4$) if the zero solution of the system of differential equations

$$u' = G_1(t,u), \quad u(t,u) = u_0 \tag{2.4}$$

is ϕ_0-equibounded, and if the zero solution of the system

$$w' = G_2(t,w), \quad w(t_0) = w_0 \tag{2.5}$$

is uniformly ϕ_0-bounded.

Then the zero solution of (1.1) is ϕ_0-equibounded.

Proof. Since E is a compact subset of C_0, there exists $\rho > 0$ such that $s_0(\rho) \supset s_0(E, \rho_0)$ for some $\rho_0 > 0$, where

$$s_0(E, \rho_0) = \{x_t \in C_0 : d(x_t, E) < \rho_0\},$$

where

$$d(x_t, E) = \inf_{y_t \in E} \parallel x_t - y_t \parallel .$$

Let $t \in \mathbb{R}^+$, $\alpha \leqslant \rho$ be given, assume that $\alpha_1 = \alpha_1(t_0, \alpha) = \max(\alpha_0, \alpha^*)$, where

$$\alpha_0 = \max[[V_1(t_0, \psi) : \psi \in \overline{s_0(\alpha) \cap E^c}]$$

and $\alpha^* \geqslant V_1(t, x_t)$, for $(t, x_t) \in J \times \partial E$.

Since the zero solution of (2.4) is ϕ_0-equibounded, given $\alpha_1 > 0$ and $t_0 \in \mathbb{R}^+$, there exists $\beta_0 = \beta_0(t, \alpha_1)$ such that for $\phi_0 \in K^*$

$$(\phi_0, r_1(t, t_0, u_0)) < \beta_0, \quad t \geqslant t_0, \tag{2.6}$$

whenever $(\phi_0, u_0) < \alpha_1$, where $r_1(t, t_0, u_0)$ is the maximal solution of (2.4).

Also, since the zero solution of (2.5) is uniformly ϕ_0-bounded, given $\alpha_2 > 0$, $t_0 \in \mathbb{R}^+$, there exists $\beta_1(\alpha_2) > 0$, such that $\phi_0 \in K_0^*$

$$(\phi_0, r_2(t, t_0, u_0)) < \beta_1(\alpha_1), \quad t \geqslant t_0, \tag{2.7}$$

provided that $(\phi_0, w_0) < \alpha_2$, where $r_2(t, t_0, w_0)$ is the maximal solution of (2.5).

Now, we choose $u_0 = V_1(t_0, \psi)$, and $\alpha_2 = a(\alpha) + \beta_0$. As $b(u) \longrightarrow \infty$ with $u \longrightarrow \infty$ we can choose $\beta = \beta(t_0, \alpha)$ such that

$$b(\beta) > \beta_1(\alpha_2). \tag{2.8}$$

Now, let $(\phi_0, \psi) \in s_0(\alpha)$, which implies that the maximal solution $x_t^*(t_0, \psi)$ satisfies $(\phi_0, x_t^*) \in s_0(\beta)$ for $t \geqslant t_0$. Suppose that this is not true, then there exists $t^* > t_0$ for the maximal solution $x_t^*(t_0, \psi)$ of (1.1), with $(\phi_0, \psi) \in s_0(\alpha)$, such that

$$(\phi_0, x_{t^*}^*(t_0, \psi)) = \beta.$$

Since $s_0(E, \rho) \subset s_0(\alpha)$, there are two possibilities to consider:

(I) $x_t(t_0, \psi) \in E^c$, for $t \in [t_0, t^*]$,

(II) there exists $t_2 \geqslant t_0$ such that

$$x_t(t_0, \psi) \in \partial E, \quad x_t(t_0, \psi) \in E^c \quad \text{for } t \in [t_0, t^*].$$

If (I) holds, then we can find $t_1 > t_0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, x_{t_1}^*) = \alpha, \quad (\phi_0, x_{t^*}^*) = \beta, \quad \text{and} \quad (\phi_0, x_t^*) \in s_0^*(\alpha), \quad t \in [t_0, t^*]. \tag{2.9}$$

Set

$$m(t) = V_1(t, x_t(t_0, \psi)) + V_2(t, x_t(t_0, \psi)), \quad t \in [t_1, t^*].$$

It is easy to obtain, from (2.3), thus from (Lakshmikantham & Leela, 1969a,b),

$$D^+ m(t) \leqslant G_2(t, m(t)), \quad t \in [t_1, t^*],$$

and thus

$$D^+(\phi_0, m(t)) \leqslant G_2(t, m(t)), \quad t \in [t_1, t^*].$$

Consequently, comparing (Lakshmikantham & Leela, 1969a, Theorem 1.4.1), we get

$$m(t) \leqslant r_2(t, t_1, m(t)), t \in [t_1, t^*],$$

where $r_2(t, t_1, w_0)$ is the maximal solution of (2.5) such that $r_2(t_1, t_1, w_0) = w_0$; therefore for $\phi_0 \in K_0^*$

$$(\phi_0, m(t)) \leqslant (\phi_0, r_2(t, t_1, m(t))), \quad t \in [t_1, t^*].$$

Thus

$$(\phi_0, V_1(t^*, x_{t^*}(t_0, \psi)) + V_2(t^*, x_{t^*}(t_0, \psi))) \leqslant (\phi_0, r_2(t^*, t_1, V_1(t_1, x_{t_1}(t_0, \psi)) + V_2(t_1, x_{t_1}(t_0, \psi)))). \tag{2.10}$$

Similarly, from (2.1) we have

$$(\phi_0, V_1(t_1, x_{t_1}(t_0, \psi))) \leqslant (\phi_0, r_1(t_1, t_0, V_1(t_0, \psi))), \tag{2.11}$$

where $r_1(t_1, t_0, u_0)$ is the maximal solution of (2.4).

From the fact that $u_0 = V_1(t_0, \psi_1) < \alpha_1$ and using (2.6) we obtain for $\phi_0 \in K_0^*$

$$(\phi_0, r_1(t_1, t_0, V_1(t_0, \psi))) \leqslant \beta_0. \tag{2.12}$$

Furthermore,

$$(\phi_0, V_2(t_1, x_{t_1}(t_0, \psi))) \leqslant a(\alpha). \tag{2.13}$$

From (2.12) and (2.13), we have

$$\begin{aligned}(\phi_0, w_0) &= (\phi_0, V_1(t_1, x_{t_1}(t_0, \psi))) + (\phi_0, V_2(t_1, x_{t_1}(t_0, \psi))) \\ &< \beta_0 + a(\alpha) = \alpha_2.\end{aligned} \tag{2.14}$$

Hence, from (2.2), (2.7), (2.8), (2.9), (2.10), (2.14), and the fact that $V_1 \geqslant 0$,

$$b(\beta) \leqslant \beta_1(\alpha_2) \leqslant b(\beta). \tag{2.15}$$

If the case (II) holds, then we again arrive at the inequality (2.10), where $t_1 > t$ satisfies (2.9). Now, we have in place of (2.11) the inequality

$$(\phi_0, V_1(t_1, x_{t_1}(t_0, \psi))) \leqslant (\phi_0, r_1(t_1, t_2, V_1(t_2, x_{t_2}(t_0, \psi)))).$$

Since $x_{t_2}(t_0, \psi) \in \partial E$ and $(\phi_0, V_1(t_2, x_{t_2}(t_0, \psi))) \leqslant \alpha^* \leqslant \alpha_1$ it follows that we get the same contradiction in (2.15). This proves that

$$(\phi_0, x_t^*(t_0, \psi)) < \beta \quad \text{for } t \geqslant t_0, \phi_0 \in K_0^*,$$

whenever $(\phi_0, \psi) < \alpha, \alpha \geqslant \rho$; the proof is complete. □

3. ϕ_0-equistability

In this section, we discuss the concept of the perturbing Liapunov functional for the ϕ_0-equistability property of the system of functional differential equations (1.1).

THEOREM 2 Suppose that there exist two functions G_1 and G_2 which are defined as in Theorem 1, and let there exist two functionals $V_1(t, x_t) \in C[J \times s_0(\rho), K]$, $V_2(t, x_t) \in C[J \times s_0(\rho) \cap s_0^c(\eta), K]$, with $V_1(t, 0) = V_2(t, 0) = G_1(t, 0) = G_2(t, 0) = 0$ such that

(A_5) $V_1(t, x_t)$ is Lipschitzian in x_t and

$$D^+(\phi_0, V_1(t, x_t)) \leqslant G_1(t, V_1(t, x_t)), \quad (t, x_t) \in J \times s_0(\rho); \tag{3.1}$$

(A_6) $V_2(t, x_t)$ is Lipschitzain in x_t and

$$b(\phi_0, x_t^*) \leqslant (\phi_0, V_2(t, x_t)) \leqslant a(\phi_0, x_t^*), \tag{3.2}$$

where $a, b \in \mathcal{K}$, $(t, x_t) \in (J \times s_0(\rho) \cap s_0^c(\eta))$, and $\phi_0 \in K_0^*$;

(A_7) for each $\phi_0 \in K_0^*$, $(t, x_t) \in (J \times s_0(\rho) \cap s_0^c(\eta))$,

$$D^+(\phi_0, V_1(t, x_t)) + D^+(\phi_0, V_2(t, x_t)) \leqslant G_2(t, V_1(t, x_t) + V_2(t, x_t));$$

(A_8) if the zero solution of the system (2.4) is ϕ_0-equistable, and if the zero solution of the system (2.5) is uniformly ϕ_0-stable.

Then the zero solution of (1.1) is ϕ_0-equistable.

Proof. From our assumption, the zero solution of (2.5) is uniformly ϕ_0-stable. Let $0 < \epsilon < \rho$; given $b(\epsilon) > 0$ and $t_0 \in \mathbb{R}^+$, there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, r_2(t, t_0, w_0)) < b(\epsilon), \quad t \geqslant t_0, \tag{3.3}$$

provided that $(\phi_0, w_0) < \delta$, where $r_2(t, t_0, w_0)$ is the maximal solution of (2.5).

From the condition (A_6), there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that

$$a(\delta_0) < \tfrac{1}{2}\delta. \tag{3.4}$$

From our assumption, the zero solution of (2.4) is ϕ_0-equistable; given $\frac{1}{2}\delta$, and $t_0 \in \mathbb{R}^+$, there exists $\delta^* = \delta^*(t_0, \epsilon) > 0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, r_1(t, t_0, u_0)) < \tfrac{1}{2}\delta, \quad t \geqslant t_0, \tag{3.5}$$

provided that $(\phi_0, u_0) < \delta^*$, $r_1(t, t_0, u_0)$ being the maximal solution of (2.4).

Following (Lakshmikantham & Leela, 1976), choose $\psi = V_1(t_0, \psi)$; since $V_1(t, x_t)$ is continuous and $V_1(t, 0) = 0$, there exists $\delta_1 > 0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, \psi) < \delta \Longrightarrow (\phi_0, x_t^*(t_0, \psi)) < \epsilon, \quad t \geqslant t_0. \tag{3.6}$$

Suppose that this is not true, then there exist $t_1, t_2 > t_0$ such that for $(\phi_0, \psi) < \delta$,

$$\left.\begin{array}{l} (\phi_0, x_{t_1}^*(t_0, \psi)) = \epsilon, \\ (\phi_0, x_{t_2}^*(t_0, \psi)) = \delta, \\ (\phi_0, x_t^*(t_0, \psi)) \in s_0(\epsilon) \cap s_0^c(\delta_0), \quad t \in [t_1, t_2]. \end{array}\right\} \tag{3.7}$$

Let $\delta_2 = \eta$, so that the condition (A_6) is ensured. Setting

$$m(t) = V_1(t, x_t(t_0, \psi)) + V_{2\cdot\eta}(t, x_t(t_0, \psi)), \quad t \in [t_1, t_0],$$

we get for $\phi_0 \in K_0^*$

$$D^+(\phi_0, m(t)) \leqslant g_2(t, m(t)), \quad t \in [t_1, t_2],$$

which yields

$$(\phi_0, V_1(t_2, x_{t_2}(t_0, \psi)) + V_{2\cdot\eta}(t_2, x_{t_2}(t_0, \psi))) \leqslant (\phi_0, r_2(t_2, t_1, V_1(t_1, x_{t_1}(t_0, \psi)) + V_{2\cdot\eta}(t_1, x_{t_1}(t_0, \psi)))),$$

where $r_2(t_1, t_1, w_0) = w_0, r_2(t_1, t_1, w_0)$ is the maximal solution of (2.5). Also, we have

$$(\phi_0, V_1(t_1, x_{t_1}(t_0, \psi))) \leqslant (\phi_0, r_1(t_1, t_0, V_1(t_0, \psi))),$$

where $r_1(t_1, t_0, u_0)$ is the maximal solution of (2.4).

By (3.5), and (3.6) we have

$$(\phi_0, V_1(t_1, x_{t_1}(t_0, \psi))) \leqslant \tfrac{1}{2}\delta. \tag{3.8}$$

From (3.4), and (3.7) we get

$$(\phi_0, V_{2\cdot\eta}(t_1, x^*_{t_1}(t_0, \psi))) < \tfrac{1}{2}\delta. \tag{3.9}$$

Thus (3.3), (3.7), (3.8), (3.9) and (A_6) yield the following contradiction:

$$\begin{aligned} b(\epsilon) &= b(\phi_0, x^*_{t_1}(t_0, \psi)) \\ &\leqslant (\phi_0, V_{2.\eta}(t_1, x^*_{t_1}(t_0, \psi))) \\ &\leqslant a(\phi_0, x^*_{t_2}(t_0, \psi)) \\ &= a(\delta) \\ &\leqslant b(\epsilon). \end{aligned}$$

Thus, the zero solution of (1.1) is ϕ_0-equistable, and the proof is complete. □

REFERENCES

AKPAN, E. P. & AKINYELE, O. (1992) On the ϕ_0-stability of comparison differential systems. *J. Math. Anal. Appl.*, **164**, 307–324.

EL-SHEIKH, M. M. A. & SOLIMAN, A. A. (1995) ϕ_0-stability criteria of nonlinear systems of differential equations. *Pan Amer. Math. J.*, **5**, 17–30.

EL-SHEIKH, M. M. A. & SOLIMAN, A. A. (2000) On stability of nonlinear systems of functional differential equations. *Appl. Math. Comput.*, **107**, 81–93.

KOKSAL, S. (1992a) Stability properties and perturbing Liapunov functions. *J. Applic. Anal.*, **43**, 99–107.

KOKSAL, S. (1992b) Boundedness properties and perturbing Liapunov functions. *Math. Methods Appl. Sci.*, **163**, 73–78.

LAKSHMIKANTHAM, V. & LEELA, S. (1969) *Differential and Integral Inequalities*, I. New York: Academic Press.

LAKSHMIKANTHAM, V. & LEELA, S. (1969) *Differential and Integral Inequalities*, II. New York: Academic Press.

LAKSHMIKANTHAM, V. & LEELA, S. (1976) On perturbing Liapunov functions. *Math. Systems Theory*, **10**, 85–90.

SOLIMAN, A. A. (2002) On perturbing Liapunov functional. *Appl. Math. Comput.*, **132**, 319–325.